Optimization Theory and Methods

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- Stag Hunt Game
- Payoff Matrix
- Strategic Form Game Definition
- Mixed Strategies
- Elimination of Dominated Strategies
- Motivation for a Nash Equilibrium

- Two hunters go hunting. Each can try to hunt a stag or a hare.
- Each stag is worth 4 units of profit and each hare is worth 1 unit of profit. A stag can be successfully hunted only if they both try for it together but hare can be hunted individually.
- If both decide to hunt a stag, they are successful in hunting it and each gets a profit of 2 units.
- If both decide to hunt a hare, each is successful in hunting one hare and each gets a profit of 1 unit.
- If one decides to hunt a stag and the other decides to hunt a hare, the one trying to hunt a stag is unsuccessful and gets 0 profit but the one trying to hunt a hare is still successful and gets 1 unit of profit.
- What happens in this case? What should happen?

- Note that situations similar to stag hunt game happen often.
- One common example is where two competing firms with identical costs and identical products decide their prices. Two alternatives are stag (high price) and hare (low price).
- If the prices for both are high then both get half the market share and both have high profits.
- If the prices for both are low then both get half the market share and both have moderate profits.
- If one has high price and the other has low price, then the high price firm gets a low market share and hence very little profit. The low price firm gets a high market share, but has to invest in more capacity to satisfy the extra demand and so earns moderate profit.

↳ Payoff Matrix

- An easy way to summarize this situation is using a payoff matrix.
  - Only works when there are only two players involved.
  - Only works when the number of possible decisions per player is finite.
- Payoff matrix is a table where each cell contains 2 numbers separated by a comma. These two numbers are the profit (also called payoff) values for each of the two decision makers (also called players).
- The first number in each cell corresponds to payoff of player 1 and second corresponds to payoff of player 2.
- Each row corresponds to one value of player 1’s decision.
- Each column corresponds to one value of player 2’s decision.

↳ Payoff Matrix of the Stag Hunt Game

		Player 2 Decision	
		Stag	Hare
Player 1 Decision	Stag	2,2	0,1
	Hare	1,0	1,1

- More formally, let $\mathbb{I}$ be the set of players. Here, $\mathbb{I} = \{1,2\}$
- Let S_i be the set of available actions for player i. Here, $S_1 = \{Stag, Hare\}$ and $S_2 = \{Stag, Hare\}$.
- Let $S = \prod_i S_i$ be the set of all possible combinations of actions by the players (i.e., all possible *strategy profiles*). Here, $S = \{(Stag, Stag), (Stag, Hare), (Hare, Stag), (Hare, Hare)\}$.
- Let $u_i: S \to \mathbb{R}$ be the payoff function for player i. Here, payoff function is succinctly described by the payoff matrix above. Note that $\mathbb{R}$ represents the set of real numbers.

- A strategic form game has 3 elements:
  - The set of players $i \in \mathbb{I}$ which is a finite set $\{1,2,\ldots I\}$.
  - The pure strategy space S_i for each player i: Same as the set of actions.
  - Payoff functions u_i that give each player i's payoff (also called utility by economists) for a given *strategy profile*. So the set of strategy profiles is the same as the set of all possible combinations of actions by the players.

- A *strategy profile* or *action profile* $s = \{s_1, s_2, \ldots, s_I\}$ is an element of the set S.
    - For example, in the stag hunt game, there are 4 strategy profiles. $(Stag, Stag), (Stag, Hare), (Hare, Stag), (Hare, Hare)$.
    - In each strategy profile, there are two components.
    - E.g. in $(Hare, Stag)$, $s_1 = Hare$, $s_2 = Stag$.
    - In general, s_1 can take any value in $S_1 = \{Stag, Hare\}$ and s_2 can take any value in $S_2 = \{Stag, Hare\}$.
    - Here $S_1 = S_2$. But in general, they can be different sets.
- $s_{-i} = [s_j]_{j \neq i}$ is the vector of actions for all players except i.
- $S_{-i} = \prod_{j \neq i} S_j$ is the set of strategy profiles for all players except i.
- $(s_i, s_{-i}) \in S$ is another way of denoting a strategy profile.

- A strategy that does not involve any randomization is called a pure strategy.
- There are situations where it is advantageous for a player to randomize. This is the motivation for having mixed strategies.
- A mixed strategy is a probability distribution over pure strategies.
- In a mixed strategy, each player's randomization is statistically independent of that of every other player.
- Mixed strategy payoffs are the expected values of the payoffs to the corresponding pure strategies.
- **Note**: In strategic form games, the terms *pure strategy* and *action* mean the same. So we use them interchangeably. But this is not the case in other forms of games (e.g., in multi-stage games that we will study in 2 classes from now).

- Let $\boldsymbol{\sigma_i}$ denote a mixed strategy of player i.
- Let $\boldsymbol{\Sigma_i}$ be the set (or space) of player i's mixed strategies, i.e., all possible values of player i's mixed strategies. So, $\sigma_i \in \Sigma_i$.
- Let $\boldsymbol{\sigma_i(s_i)}$ denote the probability assigned to a pure strategy s_i under mixed strategy σ_i.
- Let $\boldsymbol{\Sigma}$ be space of mixed strategy profiles (i.e., the set of all possible combinations of mixed strategies for all players).
- Let $\boldsymbol{\sigma}$ denote a member of set Σ. So σ is used to denote a strategy profile.
- The *support* of a mixed strategy $\boldsymbol{\sigma_i}$ is the set of pure strategies to which σ_i assigns positive probability.
- Player i's payoff to a mixed strategy profile σ is given by: $u_i(\sigma) = \sum_{s\in S}\left(\prod_{j=1}^{I} \sigma_j(s_j)\right)u_i(s)$.
- Note that pure strategy is also a type of mixed strategy!

- Consider a 2-player game with each player having 3 pure strategies. Row player strategies are U (up), M (middle), and D (down). Column player strategies are L (left), M (middle), R (right).
- Let the payoff matrix be as follows:

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

- Consider a mixed strategy profile: $\sigma_1 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $\sigma_2 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.
- So the payoffs are given by, $u_1(\sigma_1, \sigma_2) = \frac{1}{3} * \left(0 * \mathbf{4} + \frac{1}{2} * \mathbf{5} + \frac{1}{2} * \mathbf{6}\right) + \frac{1}{3} * \left(0 * \mathbf{2} + \frac{1}{2} * \mathbf{8} + \frac{1}{2} * \mathbf{3}\right) + \frac{1}{3} * \left(0 * \mathbf{3} + \frac{1}{2} * \mathbf{9} + \frac{1}{2} * \mathbf{2}\right) = \frac{11}{2}$.
- $u_2(\sigma_1, \sigma_2) = \frac{1}{3} * \left(0 * \mathbf{3} + \frac{1}{2} * \mathbf{1} + \frac{1}{2} * \mathbf{2}\right) + \frac{1}{3} * \left(0 * \mathbf{1} + \frac{1}{2} * \mathbf{4} + \frac{1}{2} * \mathbf{6}\right) + \frac{1}{3} * \left(0 * \mathbf{0} + \frac{1}{2} * \mathbf{6} + \frac{1}{2} * \mathbf{8}\right) = \frac{9}{2}$.

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

- Let us again focus on the same example:
  - In each row, the green values (player 2 payoffs when player 2 plays R) are larger than the red values (player 2 payoffs when player 2 plays M).
  - So, irrespective of how player 1 plays (i.e., irrespective of the row) R strategy is better than M strategy for player 2.
  - So, we say that strategy M is strictly dominated. So a rational player 2 will never play this strategy.

- If player 1 knows that player 2 will never play strategy M, then the only possibilities for player 2 are L or R.
- In that case, the reduced payoff matrix is as follows:

	L	R
U	4,3	6,2
M	2,1	3,6
D	3,0	2,8

- Then, for either strategy of player 2 (L or R), we see that the best strategy for player 1 is U (because $4 = u_1(U,L) > u_1(M,L) = 2$, $4 = u_1(U,L) > u_1(D,L) = 3$, $6 = u_1(U,R) > u_1(M,R) = 3$ and $6 = u_1(U,R) > u_1(D,R) = 2$).
- Finally, if player 2 knows that player 1 plays strategy U, then player 2 must play L. So by iterative elimination of dominated strategies, we conclude that the only strategy profile that survives is (U,L).

- Consider another 2-player example. Assume that player 1 has 3 pure strategies, viz., U (up), M (middle), and D (down) and player 2 has 2 pure strategies, viz., L (left) and R (right).
- Let the payoff matrix be as follows:

	L	R
U	2,0	-1,0
M	0,0	0,0
D	-1,0	2,0

- No pure strategy dominates another pure strategy (Verify!).
- But, consider a mixed strategy for player 1: $\sigma_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$.
- $u_1(\sigma_1, L) = \frac{1}{2} * 2 + \frac{1}{2} * (-1) = \frac{1}{2}$ and $u_1(\sigma_1, R) = \frac{1}{2} * (-1) + \frac{1}{2} * 2 = \frac{1}{2}$.
- So player 1's *M* strategy is dominated by σ_1 and can be eliminated.

- We saw that a pure strategy can be dominated by another pure strategy or another mixed strategy.
- The dominance can be *strict* or *weak*.
- A pure strategy s_i is ***strictly dominated*** for player i if there exists $\sigma_i{}' \in \Sigma_i$ such that, $u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.
- A pure strategy s_i is ***weakly dominated*** for player i if there exists $\sigma_i{}' \in \Sigma_i$ such that, $u_i(\sigma_i', s_{-i}) \geq u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ and $u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$ for at least one $s_{-i} \in S_{i-}$.

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1) The definitions of strictly and weakly dominated strategies provided above remain valid if we replace the last part 'for all $s_{-i} \in S_{-i}$' with 'for all $\sigma_{-i} \in \Sigma_{-i}$'.

2) When a pure strategy is dominated, all mixed strategies that contain this pure strategy in its support are also dominated.

3) It is possible to have a strictly dominated mixed strategy such that none of the pure strategies in its support are even weakly dominated.

- An example of the last property is below:
  - $\sigma_1 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ is strictly dominated by *D*.
  - Yet, neither *U* nor *M* is dominated by *D*.

	L	R
U	1,3	-2,0
M	-2,0	1,3
D	0,1	0,1

- Iterated elimination of dominant strategies sometimes yields a unique strategy profile.
- In such cases, it seems to be a reasonable way of predicting the outcome. However, how sure can we be?

	L	R
U	8,10	-100,9
D	7,6	6,5

- L dominates R. So we eliminate R. Then U dominates D. So we eliminate D. The unique outcome is (U, L). Is this realistic?
- In reality, $u_1(U, R) = -100$ is much lower than everything else.
- So player 1 might try to avoid U, especially since u_2 values for R are only 1 less than those for L.
- If U is eliminated, then player 2 will choose L. So one might argue that the only reasonable outcome is (D, L).

↳ Motivation for a Nash Equilibrium

- Unfortunately, most games are not solvable using iterated elimination of strictly dominated strategies.
- Instead, a much more useful way of finding a stable outcome of a game is to use the concept of Nash equilibrium.
- We can prove that Nash equilibrium exists for several very general types of games.
- Additionally, we can also prove that for a large subset of these games, exactly one Nash equilibrium solution exists.
- Nash equilibrium is the most popular way of predicting outcomes of game situations.
- A Nash equilibrium that predicts only pure strategy solutions is called a pure strategy Nash equilibrium.
- A Nash equilibrium that predicts mixed strategies is called a mixed strategy Nash equilibrium.

Objective：

Key Concepts：